Linear Algebra.

Vector Spaces.

Introduction

In three dimensional Euclidean space, a vector v is determined uniquely by its three components (ξ, η, λ) (relative to a definite co-ordinate system) and conversely given any ordered triple (ξ, η, λ) of real nos. there exist a vector v having ξ, η, λ as its coordinates. We shall write $v = (\xi, \eta, \lambda)$ to emphasize that v is a vector.

Similarly in the two dimensional Euclidean space, a vector has two co-ordinates and is determined uniquely by its co-ordinates.

There are three fundamental operations on vectors, namely addition, multiplication by a scalar and scalar product of two vectors. Thus if $v = (\xi, \eta, \lambda)$ and $v' = (\xi', \eta', \lambda')$ are two vectors in R^3, the three dimensional Euclidean space, α is any scalar, then we define

$$v + v' = (\xi + \xi', \eta + \eta', \lambda + \lambda')$$

$$\alpha v = (\alpha \xi, \alpha \eta, \alpha \lambda)$$

and scalar product $(v, v') = \xi\xi' + \eta\eta' + \lambda\lambda'$.

In vector space we give a treatment of vectors in a more general setting.

In vector space it is not necessary to restrict our study in ordered pairs or ordered triples, instead one may take ordered n-tuples. Secondly we do not confine ourselves to coordinates being real nos. The co-ordinates may be taken from any field.

Let us give an abstract definition of a vector space.

Definition 1

A system $(V, D, \oplus, \odot)$ where V is a non-void set, D a division ring, $\oplus$ a binary composition on V and $\odot$ is an external composition ~~later~~ i.e, $\odot$ is a mapping of $D \times V$ into V which for each element $v \in V$, $\alpha \in D$ determines an element $\alpha \odot v \in V$, is called a left vector space over D if it satisfies the following conditions.

V1) $(V, \oplus)$ is an abelian group.

V2) $\forall\ x, y \in V,\ \alpha, \beta \in D$

i) $\alpha \odot (x \oplus y) = (\alpha \odot x) \oplus (\alpha \odot y)$

ii) $(\alpha \oplus \beta) \odot x = (\alpha \odot x) \oplus (\beta \odot x)$

iii) $(\alpha\beta) \odot x = \alpha \odot (\beta \odot x)$

iv) $1 \odot x = x$

Every element of V is called a vector, every element of D is called a scalar.

In this definition if we replace $\alpha \cdot v$ by $v \cdot \alpha$ then we get a right vector space over D.

Let us restrict our study only in vector spaces over fields.

In the above definition $\oplus$ is a binary ~~compositio~~ composition on V i.e, $\oplus$ is a mapping

$$\oplus : V \times V \to V.$$

Therefore $\oplus$ assign to each order pair of elements of V to a definite element of V.

$\odot$ is an external composition of D with V.

i.e, $\odot : D \times V \to V$.

1 is the identity element of F.

For convenience we shall dispence with $\oplus$ and $\odot$ and use only + and $\cdot$ to denote both types of addition and multiplication. Again we dispence with $\cdot$ and denote $c \cdot d$ in F by cd and $c \odot \alpha$ in V by $c\alpha$.

Ex. 1 For any field F, let $V = \{(\alpha_1, \alpha_2) \mid \alpha_i \in F\}$.

Then V is a vector space over a field F under vector addition and multiplication of a vector by scalar given by

$$(\alpha_1, \alpha_2) + (\beta_1, \beta_2) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2)$$

$$c(\alpha_1, \alpha_2) = (c\alpha_1, c\alpha_2).$$

Ex 2. Let V be the set of all ordered n-tuples $\{(\alpha_1, \alpha_2, \ldots, \alpha_n) : \alpha_i \in R\}$

Let + be a composition on V, called addition defined by

$$(\alpha_1, \alpha_2 \cdots \alpha_n) + (\beta_1, \beta_2 \cdots \beta_n) = (\alpha_1+\beta_1, \alpha_2+\beta_2 + \cdots + \alpha_n+\beta_n)$$

and an external composition of R with V, called multiplication by real nos. be defined by

$$c(a_1, a_2, \ldots, a_n) = (ca_1, ca_2 \ldots, ca_n), \; c \in R$$

Then V is a real vector space and is denoted by R^n.

$(0, 0, 0 \ldots 0)$ is the null vector of R^n and is denoted by θ.

Ex 3. Let V be the additive group of polynomials in one variable over a field F. For any $\alpha \in F$ and $f(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n \in F(x)$, define $\alpha f(x) = \alpha\alpha_0 + \alpha\alpha_1 x + \cdots + \alpha\alpha_n x^n$.

Then V becomes a vector space over F.

Ex 4. Let V be the set of all continuous real valued fns. defined on the closed interval $[0, 1]$. For any $f, g \in V$ and $\alpha \in R$ $f+g$ and αf as

$$(f+g)(x) = f(x) + g(x)$$

$$(\alpha f)(x) = \alpha f(x) \quad \forall x \in [0, 1]$$

Under these operations, V becomes a vector space over R. (Since the sum of continuous functions and scalar multiple of any continuous function are continuous)

Lemma.

If V is a left vector space over F then V is also a right vector space over F and conversely.

Proof.

Since V is a left vector space over F, for each $v \in V$ and $\alpha \in F$, αv is a uniquely defined element of V such that

$$\alpha(u+v) = \alpha u + \alpha v$$
$$(\alpha+\beta)u = \alpha u + \beta u.$$
$$(\alpha\beta)u = \alpha(\beta u)$$
and $$1u = u, \forall \alpha, \beta \in F \text{ and } u, v \in V,$$

and define $v\alpha$ as αv $\forall \alpha \in F$, $v \in V$.

Then for any $\alpha, \beta \in F$, $u, v \in V$, we have

$$(u+v)\alpha = \cancel{u\alpha + v\alpha}\ \alpha(u+v) = \alpha u + \alpha v$$
$$= u\alpha + v\alpha$$
$$u(\alpha+\beta) = (\alpha+\beta)u = \alpha u + \beta u = u\alpha + u\beta$$
$$u(\alpha\beta) = (\alpha\beta)u = (\beta\alpha)u \quad \text{as } \alpha\beta = \beta\alpha$$
$$= \beta(\alpha u) = \beta(u\alpha) = (u\alpha)\beta.$$
and $u1 = 1u = u$.

Thus V is an Abelian group under addition such that for every $v \in V, \alpha \in F$, $v\alpha$ is a uniquely defined element of V satisfying the following conditions:

For any $u, v \in V$, $\alpha, \beta \in F$

$$(u+v)\alpha = u\alpha + v\alpha$$

$$u(\alpha+\beta) = u\alpha + u\beta$$

$$u(\alpha\beta) = (u\alpha)\beta$$

$$u1 = u$$

Hence V is a right vector space over F.

Similarly the converse.

This proves the lemma.

Henceforth we shall not make any distinction between left and right vector spaces and confine ourselves to vector spaces over fields.

Ex. Conside $V = \{(\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in R\}$

Define $(\alpha, \beta, \gamma) + (\alpha', \beta', \gamma') = (\alpha+\alpha', \beta+\beta', \gamma+\gamma')$

$$\lambda(\alpha, \beta, \gamma) = (\lambda\alpha, \lambda\beta, 0), \forall \alpha, \beta, \gamma, \alpha', \beta', \gamma', \lambda \in R$$

Now one can verify that all the conditions V1, V2 (i), (ii), (iii) are satisfied.

But V2 (iv) is not satisfied as $(3,2,6) \in V$, but $1(3,2,6) = (3,2,0) \neq (3,2,6)$

$\therefore V$ is not a vector space.